On adiabatic renormalization and Gauge invariant backreaction in U(1)-axion inflation

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Generation, evolution, and observations of cosmological magnetic fields
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Outline

- **Adiabatic Renormalization**
  C. Animali, P.C, G. Marozzi JCAP 05, 026 (2022), arXiv:2201.05602

- **Gauge Invariant Backreaction**
  D. Campanella Galanti, P.C, G. Marozzi, S. Santos da Costa
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Introduction to the adiabatic Renormalization

- **UV divergences:** As in the case of flat space time, observable are characterized by divergences in the deep UV.
- **New divergences:** The presence of gravity led to new divergences that are not matched by the Minkowski ones.
- **Vacuum choice:** There is not a preferred choice of the vacuum.

Minimal energy vacuum at time t_0

$$\phi(x) = \sum_k \{\mathbf{A}_k f_k(x) + \mathbf{A}_k^\dagger f_k^*(x)\}\,, \qquad \mathbf{A}_k \left|0\right\rangle = 0$$

At $t > t_0$ non trivial (gravity) background mixes positives and negatives modes:

$$g(x) = \alpha_k f(x) + \beta_k f^*(x)\ , \qquad |\alpha_k^2 - \beta_k^2| = 1$$

the field can be represented as a new combination of mode functions such that:

$$\phi(x) = \sum_k \{\mathbf{B}_x g_k(x) + \mathbf{B}_k^\dagger g_k^*(x)\}, \qquad \mathbf{A}_k = \alpha_k \mathbf{B}_k + \beta_k \mathbf{B}_{-k}^\dagger$$

the "vacuum"-state is not anymore empty

$$\left\langle 0\right| \mathbf{A}_k^\dagger \mathbf{A}_k \left|0\right\rangle \neq 0 = N_k = |\beta_k|^2$$

Physical request:
particles should not be created when the energy of a single particle is larger w.r.t. the energy scale of the spacetime.

$$\frac{k^2}{a^2(t)} + m^2 > \left(\frac{\dot{a}}{a}\right)^2, \; \frac{\ddot{a}}{a} \qquad \Rightarrow \;\; N_k \sim \text{const.}$$

the particle content should not change if the change rate of $a(t)$ is adiabatic.

Adiabatic vacuum:
the vacuum that minimizes the creation of particle due to the presence of a time-dependent metric.

Scalar Field

$$\mathcal{L} = \frac{1}{2}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2 - \xi R\phi^2\right)$$

Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\,\mathrm{d}\mathbf{x}^2$$

Equation of motion: $\left(\Box + m^2 + \xi R\right)\phi = 0$

standard quantization: $\phi(x) = \sum_{\mathbf{k}} \{A_{\mathbf{k}} f_{\mathbf{k}}(x) + A_{\mathbf{k}}^\dagger f_{\mathbf{k}}^*(x)\}$

$A_{\mathbf{k}}^\dagger$ and $A_{\mathbf{k}}$ creation and annihilation operators

mode function: $f_{\mathbf{k}} = (2V)^{-1/2} a(t)^{-3/2} h_k(t) e^{i\mathbf{k}\cdot\mathbf{x}}$

the rescaled mode function $h_k(t)$ satisfies the equation

$$\ddot{h}_k + \Omega_k^2(t)\, h_k = 0$$

formally solved by the Wentzel-Kramer-Brillouin (WKB) approximation

$$h_k(t) = \frac{1}{\sqrt{2W_k(t)}} e^{-i\int W_k(t')\mathrm{d}t'}$$

inserting the WKB ansatz into the equation of motion

$$W_k(t)^2 = \Omega_k(t)^2 - \left(\frac{\ddot{W}_k(t)}{2W_k(t)} - \frac{3\dot{W}_k(t)^2}{4W_k(t)^2}\right)$$

Adiabatic condition: slowly changes in time

$$\left|\frac{\dot{W}}{W^2}\right| \ll 1$$

introducing an adiabatic parameter $\epsilon \ll 1$

$$\partial_t \to \epsilon \partial_t$$

solution for $W_k(t)$ as a power series in time derivatives

$$W_k(t) = W_k^{(0)}(t) + \epsilon\, W_k^{(1)}(t) + \cdots + \epsilon^n\, W_k^{(n)}(t)$$

Adiabatic renormalization prescription:

- evaluate expectation values w.r.t. the adiabatic vacuum
- mode functions are given in terms of WKB ansatz.
- expand up to the adiabatic order that matches divergence of the operator.
- subtract the adiabatic term from the bare quantity.

A problematic example: Axion-gauge fields

Pseudo-scalar inflaton field ϕ coupled to $U(1)$ gauge field A_μ

$$\mathcal{L} = -\frac{1}{2}(\nabla\phi)^2 - V(\phi) - \frac{1}{4}(F^{\mu\nu})^2 - \frac{g\phi}{4}F^{\mu\nu}\tilde{F}_{\mu\nu}$$

- Due to the coupling with the inflaton field ϕ, quantum fluctuations of the gauge field A_μ are amplified.

- **Backreaction**:

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = g\langle \mathbf{E}\cdot\mathbf{B}\rangle$$

$$H^2 = \frac{1}{3M_p^2}\left[\frac{\dot{\phi}^2}{2} + V(\phi) + \frac{\langle \mathbf{E}^2 + \mathbf{B}^2\rangle}{2}\right]$$

$$\dot{H} = -\frac{1}{2M_p^2}\left[\dot{\phi}^2 + \frac{2}{3}\langle \mathbf{E}^2 + \mathbf{B}^2\rangle\right]$$

Energy Density

$$\frac{\langle \mathbf{E}^2 + \mathbf{B}^2\rangle}{2} = \int^\Lambda \frac{\mathrm{d}k k^2}{(2\pi)^2 a(\tau)^4}\left[|A'_+|^2 + |A'_-|^2 + k^2\left(|A_+|^2 + |A_-|^2\right)\right]$$

Helicity Integral

$$\langle \mathbf{E}\cdot\mathbf{B}\rangle = -\int^\Lambda \frac{\mathrm{d}k k^3}{(2\pi)^2 a(\tau)^4}\frac{\partial}{\partial\tau}\left(|A_+|^2 - |A_-|^2\right)$$

The fourier mode functions $A_\pm$ satisfy the EOM:

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} A_\pm(\tau,k) + \left(k^2 \mp k g \phi'\right) A_\pm(\tau,k) = 0$$

assuming de Sitter: $a(\tau) = -1/(H\tau),\ H = \text{const.}$

$$\xi \equiv g\phi'/(2a(\tau)H) = g\dot{\phi}/(2H) \qquad \boxed{A_\pm(\tau,k) = \frac{1}{\sqrt{2k}} e^{\pm\pi\xi/2} W_{\pm i\xi,\frac{1}{2}}(-2ik\tau)}$$

- **Divergences:** $\quad \frac{\langle \mathbf{E}^2+\mathbf{B}^2\rangle}{2} \supset \Lambda^4,\ \Lambda^2, \log\Lambda,$

$$\langle \mathbf{E}\cdot\mathbf{B}\rangle \supset \Lambda^2,\ \ \log\Lambda$$

- $\Lambda^4,\ \Lambda^2$ and $\log[\Lambda]$ UV divergences for the energy density.
- Λ^2 and $\log[\Lambda]$ UV divergences for the helicity integral.
- well-behaved in the infrared

Renormalization

For each polarization $\lambda = \pm$: $\boxed{A_\lambda^{\mathsf{WKB}}(k,\tau) = \frac{1}{\sqrt{2\Omega_\lambda(k,\tau)}} e^{-i\int \Omega_\lambda(k,\tau')\mathrm{d}\tau'}}$

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} A_\pm^{\mathsf{WKB}}(\tau,k) + \left(k^2 \mp gk\phi' + \frac{m^2}{H^2\tau^2}\right) A_\pm^{\mathsf{WKB}}(\tau,k) = 0$$

$$\Downarrow$$

$$\boxed{\Omega_\lambda^2(k,\tau) = \bar{\Omega}_\lambda^2(k,\tau) + \frac{3}{4}\left(\frac{\Omega_\lambda'(k,\tau)}{\Omega_\lambda(k,\tau)}\right)^2 - \frac{1}{2}\frac{\Omega_\lambda''(k,\tau)}{\Omega_\lambda(k,\tau)}}$$

- **Adiabatic condition**: slowly changes in time: $\left|\frac{\dot{\Omega}}{\Omega^2}\right| \ll 1$

$$\epsilon \ll 1 : \partial_t \to \epsilon\partial_t$$

$$\Omega_k(t) = \Omega_k^{(0)}(t) + \epsilon\,\Omega_k^{(1)}(t) + \cdots + \epsilon^n\,\Omega_k^{(n)}(t)$$

Standard adiabatic regularization: ill defined for $m \to 0$

$$\frac{1}{2}\langle \mathbf{E}^2 + \mathbf{B}^2 \rangle_{\mathsf{ad}} = \int_0^{a(\tau)\Lambda} \frac{\mathrm{d}k k^2}{(2\pi)^2 a(\tau)^4} (\cdots)_{\mathsf{ad}}^{n=4} \supset \Lambda^4,\ \Lambda^2, \log\Lambda, \log m$$

$$\langle \mathbf{E} \cdot \mathbf{B} \rangle_{\mathsf{ad}} = -\int_0^{a(\tau)\Lambda} \frac{\mathrm{d}k k^3}{(2\pi)^2 a(\tau)^4} \frac{\partial}{\partial\tau} (\cdots)_{\mathsf{ad}}^{n=4} \supset \Lambda^2, \log\Lambda, \log m$$

- The standard adiabatic renormalization correctly removes the divergences in the UV, introducing **unphysical IR divergences**.

Issues and Motivations: the need of a IR cut off

- Adiabatic renormalization concerns the UV divergences
- WKB is well defined for modes that feel small curvature
- Good approximation for sub-horizon modes

$\langle T_{\mu\nu}\rangle_{\mathsf{ad}}$ will have the following general structure:

$$\langle T\rangle_{\mathsf{ad}}^{(n>4)} = H^4 \sum_{n>4} \left(c_n \left(\frac{H}{m}\right)^{n-4} + c'_n \left(\frac{H}{\Lambda}\right)^{n-4} \right)$$

- **deep UV**:when $\Lambda \to \infty$, the higher order terms go to zero and we can truncate the series at the fourth adiabatic order, which is indeed the order needed to remove the UV divergences.

- **IR regime** the IR regime produces higher order terms involving m which are increasingly relevant for $m \to 0$.

New adiabatic regularization

We suggest that the procedure of adiabatic regularization should be always performed on a proper domain which excludes the IR tail of the spectrum.

- the adiabatic subtraction should be considered only up to a comoving IR cut-off $c = \beta a(t) H(t)$.
- the coefficient β, should be determined by a proper physical prescription.

$$\boxed{\begin{aligned} \frac{1}{2}\left\langle \mathbf{E}^2+\mathbf{B}^2\right\rangle_{\mathsf{ad}} &= \int_{\beta a(\tau)H}^{a(\tau)\Lambda} \frac{\mathrm{d}k k^2}{(2\pi)^2 a(\tau)^4}\left(\cdots\right)_{\mathsf{ad}}^{n=4} \\ \left\langle \mathbf{E}\cdot\mathbf{B}\right\rangle_{\mathsf{ad}} &= -\int_{\beta a(\tau)H}^{a(\tau)\Lambda} \frac{\mathrm{d}k k^3}{(2\pi)^2 a(\tau)^4}\frac{\partial}{\partial\tau}\left(\cdots\right)_{\mathsf{ad}}^{n=4} \end{aligned}}$$

$$\begin{aligned} \frac{1}{2}\left\langle \mathbf{E}^2+\mathbf{B}^2\right\rangle_{\mathsf{ad}}^{c=\beta H a(\tau)} = &\ \frac{\Lambda^4}{8\pi^2}+\frac{H^2\Lambda^2\xi^2}{8\pi^2}+\frac{3H^4\xi^2(5\xi^2-1)\log\left(2\Lambda/H\right)}{16\pi^2} \\ &-\frac{\beta^4H^4}{8\pi^2}-\frac{\beta^2H^4\xi^2}{8\pi^2}-\frac{3H^4\xi^2(5\xi^2-1)\log\left(2\beta\right)}{16\pi^2} \end{aligned}$$

$$\begin{aligned} \left\langle \mathbf{E}\cdot\mathbf{B}\right\rangle_{\mathsf{ad}}^{c=\beta H a(\tau)} = &-\frac{H^2\Lambda^2\xi}{8\pi^2}-\frac{3H^4\xi(5\xi^2-1)\log\left(2\Lambda/H\right)}{8\pi^2} \\ &+\frac{\beta^2H^4\xi}{8\pi^2}+\frac{3H^4\xi(5\xi^2-1)\log\left(2\beta\right)}{8\pi^2} \end{aligned}$$

How to fix the scheme
Conformal anomaly

- In the conformal limit, a proper renormalization scheme should provide the conformal anomaly induced by quantum effects.

- When at the classical level $T^\mu{}_\mu = 0$

$$\langle T^\mu{}_\mu \rangle_{\mathsf{phys}} = -\langle T^\mu{}_\mu \rangle_{\mathsf{reg}}$$

  where $\langle T^\mu{}_\mu \rangle_{\mathsf{reg}}$ is the trace contribution to the energy-momentum tensor given by the particular renormalization method applied.

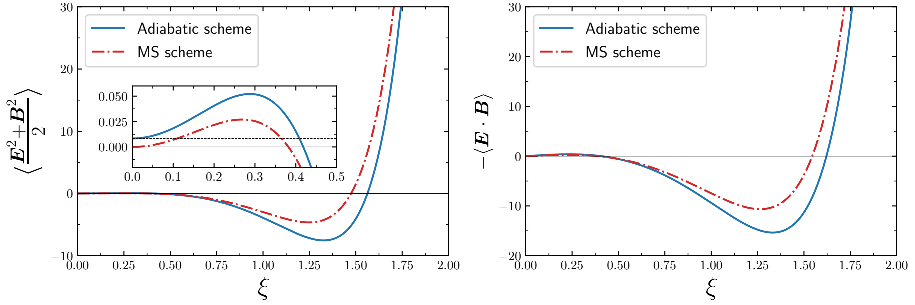
- The two helicities of the mode functions $A_\pm$ are equivalent to two conformally coupled massless scalar fields for $\xi = 0$

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}A_\pm + \left(k^2 \pm \frac{2k\xi}{\tau} + \frac{m^2}{H^2\tau^2}\right)A_\pm = 0 \to \boxed{\left(\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + k^2\right)A_\pm = 0}$$

$$\lim_{\xi\to 0,\, m\to 0}\langle T^0{}_0\rangle_{\mathsf{ad}} = \lim_{\xi\to 0,\; m\to 0}\frac{\langle \mathbf{E}^2 + \mathbf{B}^2\rangle_{\mathsf{ad}}^{c=\beta H a(\tau)}}{2} = -\frac{\beta^4 H^4}{8\pi^2}$$

this term should reproduce the expected value of the anomaly

$$\frac{\beta^4 H^4}{8\pi^2} = \frac{H^4}{480\pi^2} \implies \boxed{\beta = \frac{1}{\sqrt{2}\times 15^{1/4}} \approx 0.359}$$



Comparison between the new adiabatic scheme and the minimal subtraction scheme (MS).

- This is a physically motivated prescription that is able to fix univocally the renormalization scheme.
- We are able to obtain finite results for the averaged energy density and helicity of gauge fields.

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♦ First order perturbation gauge fields : $\boxed{A(t,x) = A^{(1)}(t,\mathbf{x})}$

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{g\phi}{4}F^{\mu\nu}\tilde{F}_{\mu\nu}\,,$$

$$\boxed{\begin{aligned} \phi(t,\mathbf{x}) &= \phi^{(0)}(t) + \varphi(t,\mathbf{x}) + \varphi^{(2)}(t,\mathbf{x}) \\ g_{\mu\nu}(t,\mathbf{x}) &= g^{(0)}_{\mu\nu}(t) + \delta g^{(1)}_{\mu\nu}(t,\mathbf{x}) + \delta g^{(2)}_{\mu\nu}(t,\mathbf{x}) \end{aligned}}$$

Perturbed metric around a FLRW:

$$g_{00} = -1 - 2\alpha - 2\alpha^{(2)}$$
$$g_{i0} = -\frac{a}{2}(\beta_{,\,i} + B_i) - \frac{a}{2}\left(\beta^{(2)}_{,\,i} + B^{(2)}_i\right)$$
$$g_{ij} = a^2\left[\delta_{ij}\left(1 - 2\psi - 2\psi^{(2)}\right) + D_{ij}\left(E + E^{(2)}\right) + \frac{1}{2}(\chi_{i,j} + \chi_{j,i} + h_{ij}) + \frac{1}{2}\left(\chi^{(2)}_{i,j} + \chi^{(2)}_{j,i} + h^{(2)}_{ij}\right)\right]$$

$$\boxed{G^\mu_\nu = 8\pi G T^\mu_\nu}$$

Einstein tensor: $G^\mu_\nu = G^{\mu(0)}_\nu + \delta G^{\mu(1)}_\nu + \delta G^{\mu(2)}_\nu$

Energy-momentum tensor: $T^\mu_\nu = T^{\mu(0)}_\nu + \delta T^{\mu(1)}_\nu + \delta T^{\mu(2)}_\nu$

- First order perturbations are decoupled
- Second order in uniform curvature gauge (UCG) ($\psi = E = 0$)

<u>Scalar sector</u>

$$\alpha^{(2)} = 4\pi G \frac{\dot{\phi}}{H} \varphi^{(2)} + s + \Gamma$$

$$\frac{H}{a} \nabla^2 \beta^{(2)} = 8\pi G \frac{\dot{\phi}^2}{H} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{H}{\dot{\phi}} \varphi^{(2)} \right) - (Q + \Sigma) + 16\pi G V (s + \Gamma)$$

$$\ddot{\varphi}^{(2)} + 3H\dot{\varphi}^{(2)} - \frac{\nabla^2 \varphi^{(2)}}{a^2} + \left[V_{\phi\phi} + 2\frac{\dot{H}}{H} \left(3H - \frac{\dot{H}}{H} + 2\frac{\ddot{\phi}}{\dot{\phi}} \right) \right] \varphi^{(2)} = D$$

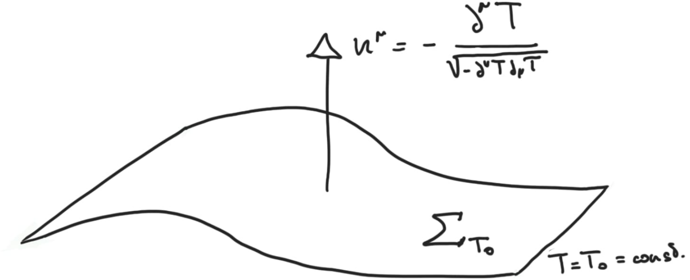
where

$$s \supset (\varphi\,,\, \alpha, \beta)\ , \quad Q \supset \varphi, (\alpha\,, \beta), \quad (\Gamma\,,\ \Sigma) \supset (E\,, \mathbf{B})\,, \quad D \supset (\mathbf{E}\,, \mathbf{B}, \,, \varphi\,,\, \alpha, \beta)$$

A physical gauge invariant observable is can be defined as

$$\langle S\rangle_{T_0} = \frac{\langle\sqrt{|\overline{\gamma}(t_0,\mathbf{x})|}\ \overline{S}(t_0,\mathbf{x})\rangle}{\langle\sqrt{|\overline{\gamma}(t_0,\mathbf{x})|}\rangle}$$

$$\bar{x}^{\mu} = (\bar{t},\mathbf{x}) :\ T(\bar{x}^{\mu}) = T(\bar{t})\,, T_0 = T(\bar{t} = t_0)$$



$$\Theta = \nabla_{\mu} n^{\mu}$$

$$H^2_{\text{eff}} \equiv \left(\frac{1}{a_{eff}}\frac{\partial a_{eff}}{\partial T_0}\right)^2 = \frac{1}{9}\left\langle \frac{\Theta}{\sqrt{-\partial^{\mu}T\partial_{\mu}T}}\right\rangle^2_{T_0}$$

G. Marozzi, M. Gasperini and G. Veneziano, JCAP 03, 011 (2009), arXiv:0901.1303

G. Marozzi, M. Gasperini and G. Veneziano, JCAP 02, 009 (2010), arXiv:0912.3244

- Natural observer $\bar{\varphi}^{(2)} = 0$ to fix $\bar{x}$

$$H^2_{\text{eff}} = \left(\frac{1}{a_{eff}}\frac{\partial a_{eff}}{\partial A_0}\right)^2 = H^2\left[1 + \frac{2}{H}\langle\bar{\psi}\dot{\bar{\psi}}\rangle - \frac{2}{H}\langle\dot{\bar{\psi}}^{(2)}\rangle\right].$$

- back to the uniform curvature gauge (UCG) in the long wavelength limit

$$\bar{\psi}^{(2)} = \frac{H}{\dot{\phi}}\varphi^{(2)}$$

$$H^2_{\text{eff}} = \simeq H^2\left[1 - \frac{2}{\dot{\phi}}\left(\frac{\dot{H}}{H}\langle\varphi^{(2)}\rangle + \langle\dot{\varphi}^{(2)}\rangle\right)\right]$$

at leading order in slow-roll

$$\boxed{H^2_{\text{eff}} \sim H^2\left[1 + \frac{1}{3H^2}\left(\frac{\langle\mathbf{E}^2 + \mathbf{B}^2\rangle}{2} - \frac{g^2}{\xi}\langle\mathbf{E}\cdot\mathbf{B}\rangle\right)\right]}$$

Summary

- Adiabatic renormalization is a powerful renormalization scheme to regularize UV divergences.
- Should be truncated up to an IR cut-off proportional to the horizon size.
- This cut-off should be fixed by a proper physical prescription.
- Work in progress: taking into account metric perturbations can have a non trivial impact on the backreaction.

Thank you for the attention