Stochastic effects and Primordial Magnetogenesis

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Thursday, 9 May 2024

See talks by:

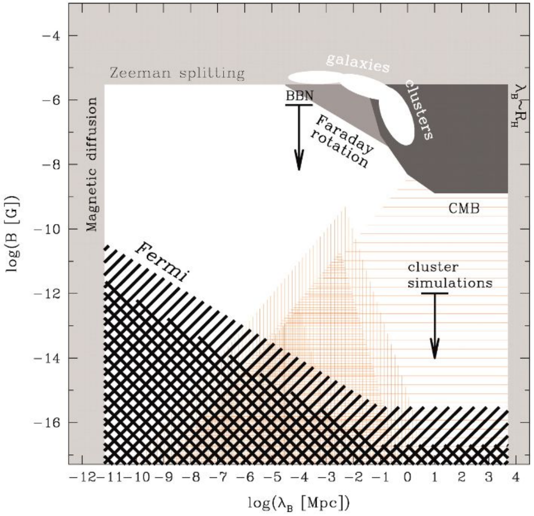
Lorenzo, Dani,
Ruth, Juan, Tina,
Tomohiro, Bharat,...

In the past eight days, I've heard some amazing talks. While it's challenging to introduce entirely new content, I'm excited to present some new questions …

Magnetic Fields in the Sky

Observed Magnetic Fields on scales L:

- ❖ **Galaxies:** $B \simeq 50\ \mu G$ @ $L < 1$ kpc;
  $B \simeq 5 - 10\ \mu G$ @ $L \sim 10$ kpc
- ❖ **Clusters:** $B \simeq 1\ \mu G$ @ $L \sim 1$ Mpc
- ❖ **Superclusters:** $B < 10^{-2} - 10^{-3}\ \mu G$ @ $L \sim 1 - 50$ Mpc
- ❖ **CMB:** $B < 10^{-3} - 10^{-5}\ \mu G$ @ $L > 100$ Mpc
- ❖ **MF inside Voids** $B \simeq 10^{-16} - 10^{-18}$ G @ $L \simeq 1$ Mpc
- ❖ **Primordial Nucleosynthesis:** $B < 10^{11}$ G @ $T = 10^{9}$ K



A. Neronov & E. Vovk, 2010

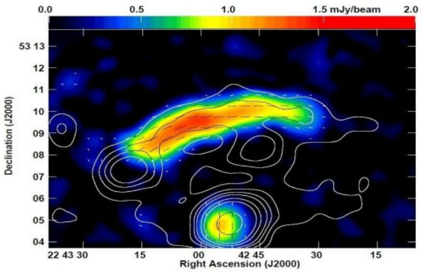


Figure: Kierdorf et al.: (2017): Sausage galaxy cluster (CIZA J2242+53)

Figure: NASA: Cigar Galaxy. A Magnetic Universe

Origins:

- ❖ **Astrophysical Processes:**
  - ➢ **dynamo mechanism:**

    Primordial Seed $B \simeq 10^{-20}$ G @ $L \simeq 1$ Mpc >>>>>>>>>>>>>>>> $B \simeq \mu G$ @ $L \sim 1$ kpc
- ❖ **Cosmological Mechanisms:**
  - ➢ **Phase transition, Topological defects, …**
  - ➢ **Inflation (can create large coherence lengths!)**

INFLATION: Formal Approach

Inflation (quantum) field:

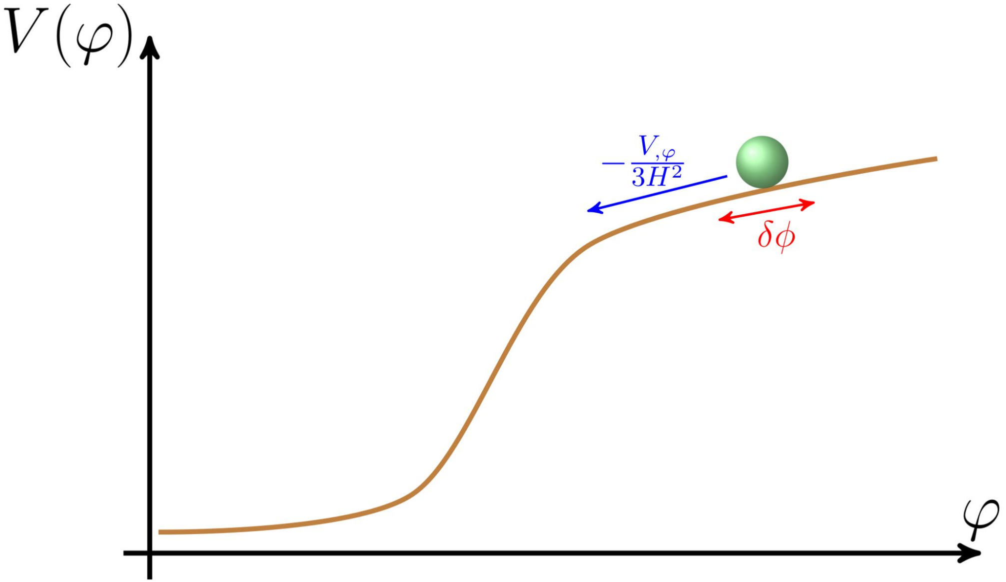
$$\hat{\Phi} = \varphi(t) + \frac{\delta\phi(t,x)}{a(t)}$$

Starobinsky 80,
Guth 81,
Linde 82,
...

Background field —>>> accelerated expansion >> Homogeneous

Slow-roll Eqs.

$$3M_{\rm Pl}^2 H^2 \simeq V(\varphi)$$
$$3H\dot{\varphi} \simeq -V_\varphi$$



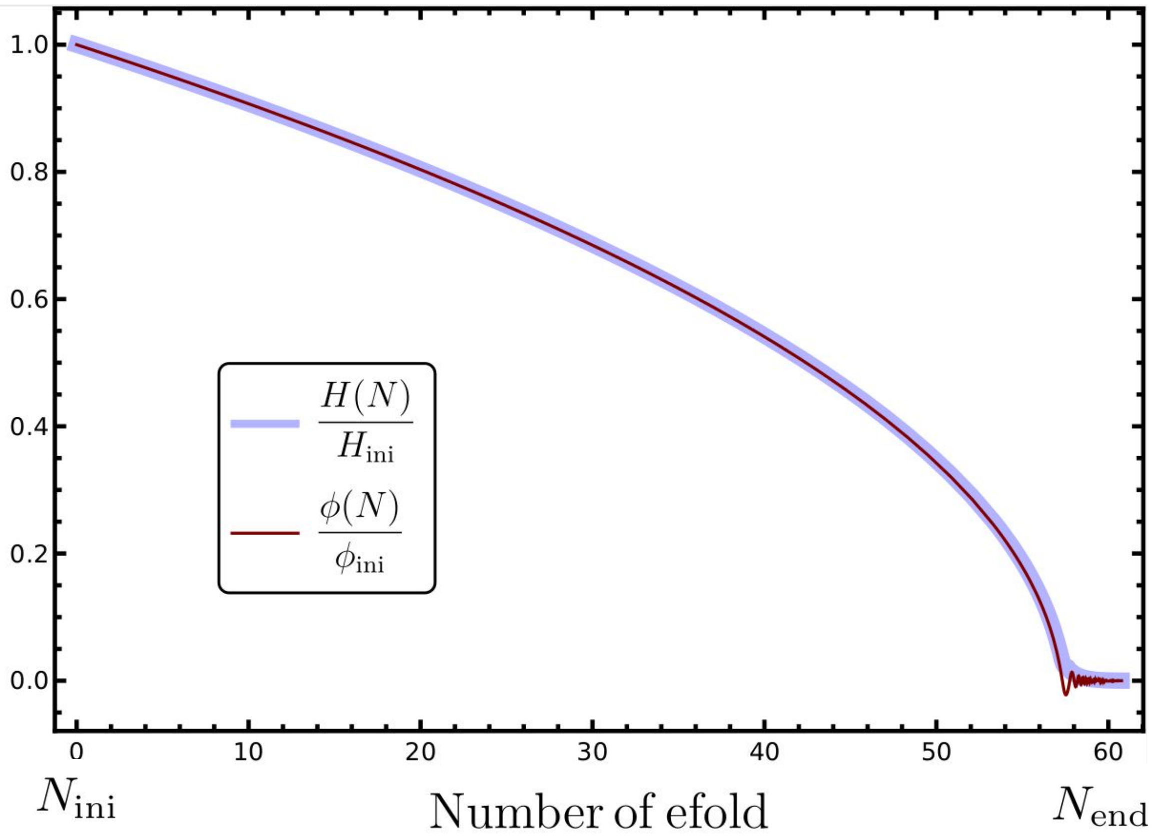
Small quantum perturbations >> Structure Formations

$$\delta\phi_k'' + \left(k^2 - \frac{2}{\tau^2}\right)\delta\phi_k \simeq 0$$

Negative mass: production

$2^{\rm nd}$ Slow-roll condition: $V_{\phi\phi} \ll H^2$

Single (Slow-roll) Inflation



Action:

$$\mathcal{S} = \int \mathrm{d}^4x \, \sqrt{g}\left[\frac{M_{\rm Pl}^2}{2}R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2\right]$$

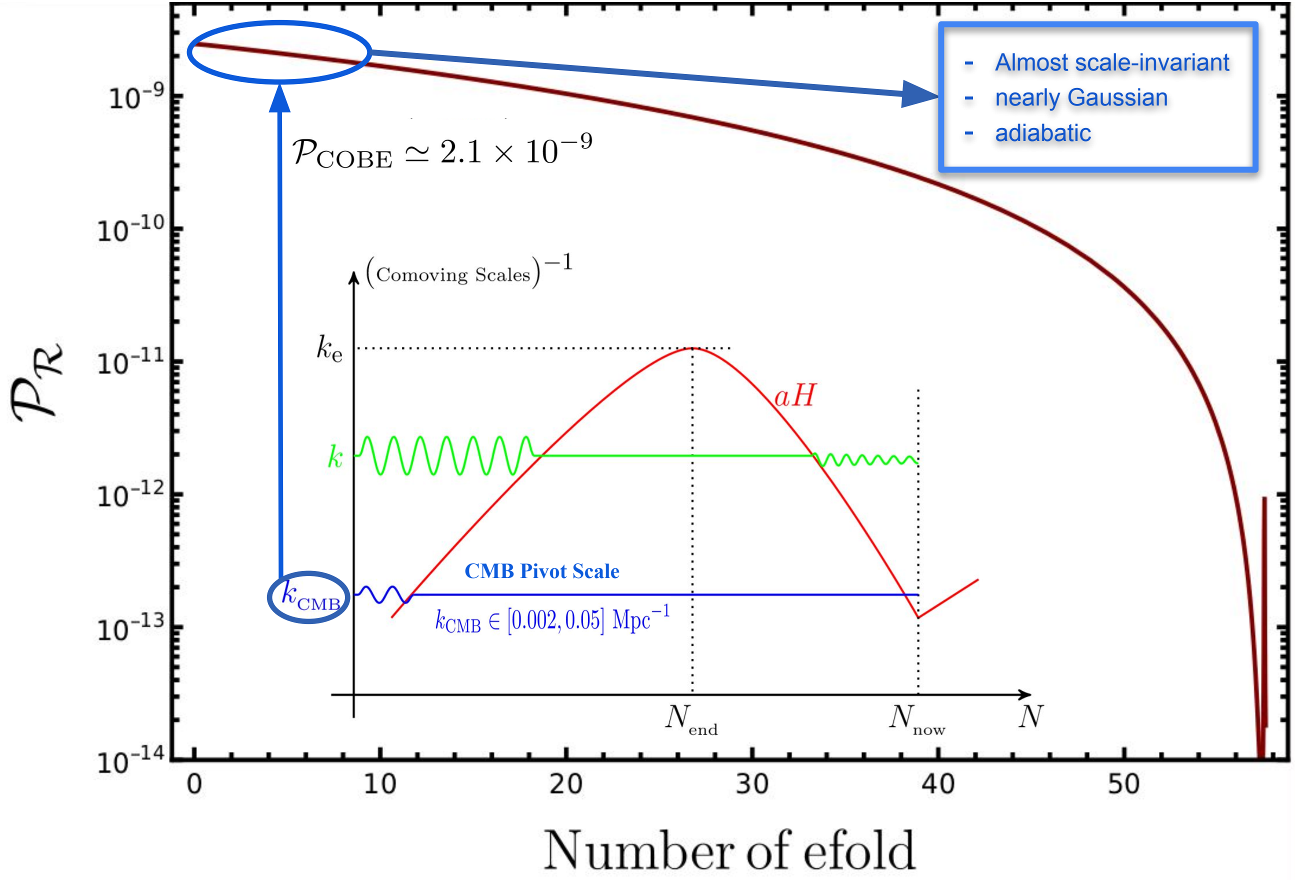
Slow-roll Eqs.

$$3M_{\rm Pl}^2 H^2 \simeq \frac{1}{2}m^2\phi^2$$

$$3H\dot{\phi} \simeq -m^2\phi$$

$$\boxed{N_{\rm end} \simeq \frac{1}{4}\left(\frac{\phi_{\rm ini}}{M_{\rm Pl}}\right)^2}$$

Scalar Perturbations



$$\delta\phi_k'' + \left(k^2 - \frac{2}{\tau^2}\right)\delta\phi_k \simeq 0$$

Spatially-flat gauge

$$\mathcal{R}_k \equiv \frac{H}{a\ \dot{\varphi}}\delta\phi_k$$

- **Gauge invariant**
- **Conserved outside Horizon**

$$k \ll aH \rightarrow \dot{\mathcal{R}}_k \simeq 0$$

Electromagnetic Fields: during inflation

$$\mathcal{S}_{\mathrm{Maxwell}} = \int \mathrm{d}^4x\sqrt{g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)$$

dS: conformally at Universe

$$\hat{A} = \mathcal{A}(t) + \delta A(t, x)$$

Planck: EM backgrounds are tightly constrained by CMB dipole anisotropies!

Maxwell is a free theory on Minkowski

$$\delta A_k'' + k^2 \delta A_k = 0$$

EM fluctuations can not be enhanced in Maxwell theory in dS background!

Electromagnetic Fields: during inflation

Breaking the Conformal Invariance (preventing the dilution of EM field during inflation)
A simple way: introduce an interaction between the EM field and the scalar field or with the curvature scalars!

$$S = \int \mathrm{d}^4x \sqrt{-g}\left[\frac{M_P^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) + \mathcal{L}_{\mathrm{EM}}\right]$$

Motivations for $\mathcal{L}_{\mathrm{EM}}$
- Is Inflaton ϕ alone?
- Statistical anisotropy — CMB: $|g_*| \leq 10^{-2}$
- Primordial magnetic field — Blazars: $B_{\mathrm{Mpc}} \sim 10^{-16}\mathrm{G}$
- Galactic magnetic field — Milky Way: $B_{\mathrm{MW}} \sim \mu\mathrm{G}$

Model: Ratra-like coupling f^2F^2+ Axion-like $F\tilde{F}$ coupling:

$$\mathcal{L}_{\mathrm{EM}} \supseteq -\frac{f^2(\phi)}{4}F_{\mu\nu}F^{\mu\nu} - \frac{I^2(\phi)}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}$$

Canonically normalized field

$$v_k \equiv f\delta A_k$$

$$v_{k,\lambda}'' + \left(k^2 - 2\lambda\frac{I\ I'}{f^2}k - \frac{f''}{f}\right)v_{k,\lambda} \simeq 0$$

Primordial Magnetic Fields, Demozzi et al 09

$$S = \int \mathrm{d}^4x \sqrt{-g}\left[\frac{M_P^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - \frac{f^2(\phi)}{4}F_{\mu\nu}F^{\mu\nu}\right]$$

$$f(\phi(\eta)) = f(\eta) = f_{\mathrm{end}}\left(\frac{\eta}{\eta_{\mathrm{end}}}\right)^n, \eta \in (-\infty, 0), \begin{cases} n < 0 & \text{strong coupling regime} \\ n > 0 & \text{weak coupling regime}. \end{cases}$$

See Bharat's talk

Slow-roll limit: $3M_P^2H^2 \simeq V(\phi) + \frac{1}{2}(E^2 + B^2)$,

Back-reaction parameter: $R \equiv \frac{E^2 + B^2}{6M_P^2H^2} \ll 1$,

$$E_i = -f\frac{\partial_\eta A_i}{a^2}$$

$$B_i = f\frac{\epsilon_{ijk}\partial_j A_k}{a^2}$$

Simplifying assumptions:
- Instant reheating
- No Faraday's induction
- …

$$\Rightarrow \quad B_{\mathrm{now}} = 2.5 \times 10^{-57}\left(\frac{r_{\mathrm{t}}}{0.01}\right)^{-\frac{1}{2}} B_{\mathrm{end}}$$

See Kohei's talk: Baryon Isocurvature Problem!

Primordial Magnetic Fields, Demozzi et al 09

$$10^{-9}G \gtrsim B_{\rm obs} \gtrsim 10^{-16}G \times \begin{cases} 1 & \lambda_{\rm B} \gtrsim 1{\rm Mpc} \\ \sqrt{\dfrac{1{\rm Mpc}}{\lambda_{\rm B}}} & \lambda_{\rm B} \lesssim 1{\rm Mpc} \end{cases}$$

CMB data (lower bound: $10^{-9}G$); Blazars data (upper bound: $10^{-16}G$)

Weak coupling regime

$$B_{\rm end} \simeq \frac{H^2}{2\pi}\left(\frac{\lambda_{\rm ph}}{H^{-1}}\right)^{n-3}$$

n=2: $B_{\rm now} \simeq 10^{-35}G$

n=2.2: $B_{\rm now} \simeq 10^{-30}G$

n=3: $B_{\rm now} \simeq 10^{-11}G$ $\star$

$\star$: Electric back-reaction problem

No back-reaction limit: $n \leq 2.2$

Strong coupling regime

$$B_{\rm end} \simeq \frac{H^2}{2\pi}\left(\frac{\lambda_{\rm ph}}{H^{-1}}\right)^{-n-2}$$

n=-3: $B_{\rm now} \simeq 10^{12}G!$ $\star$

n=-2.2: $B_{\rm now} \simeq 10^{-7}G$

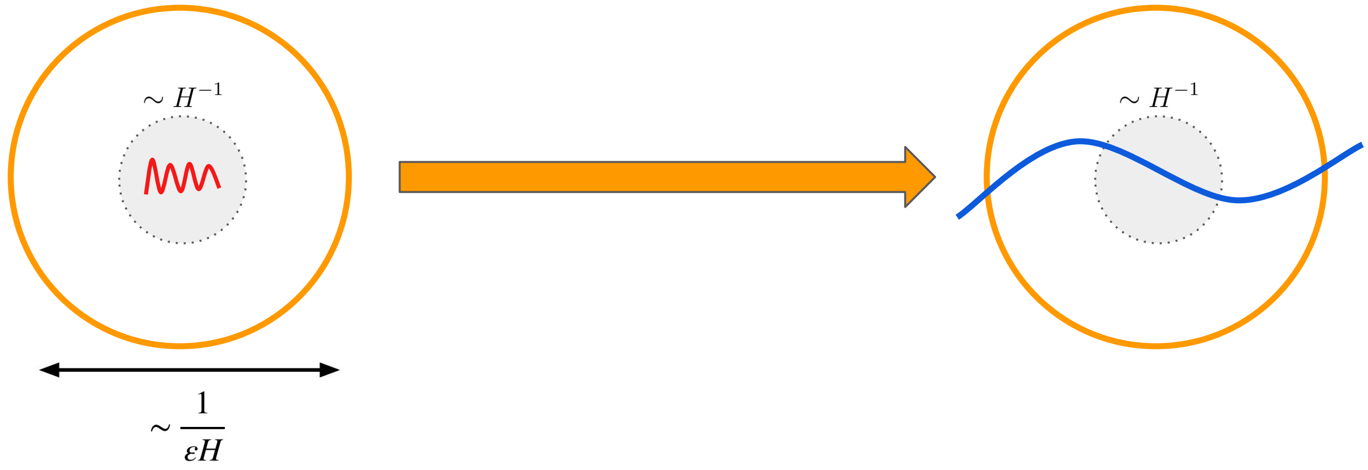
n=-2: $B_{\rm now} \simeq 10^{-11}G$

$\star$: Magnetic back-reaction problem

No back-reaction limit: $n \geq -2.2$

INFLATION: Stochastic Approach

In de Sitter spacetime, (quantum) short wavelength modes (UV) are stretched to (classical) long wavelength modes (IR)



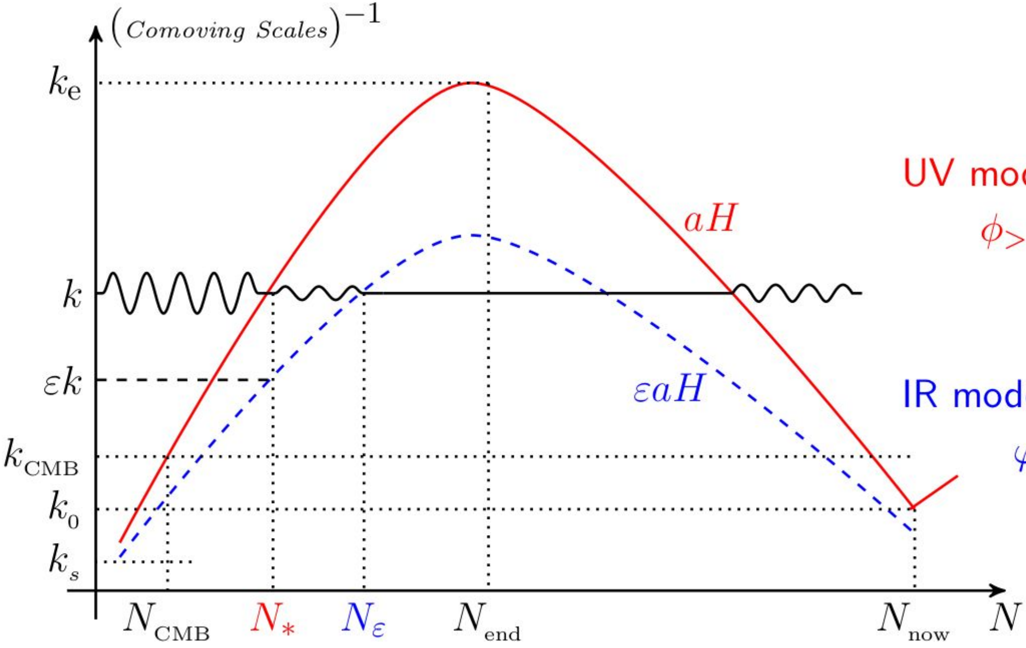
- Quantum fluctuations experience a quantum-to-classical transition.
- The IR modes evolve as the background.
- The UV modes are contribute as a quantum noise.
- We focus on the IR dynamics.

Starobinsky (82, 86),
Nambu, Sasaki (88),
Fujita et al. (13)
Vennin, Starobinsky (15)
Grain, Vennin (17)
…

INFLATION: Stochastic Approach

$$\hat{\Phi} = \varphi + \phi_{>}$$

Coarse-graining: $W_H(k,t) = \Theta(k - \varepsilon a H)$



UV modes:

$\phi_{>}$: $k > \varepsilon a H$

contains all short-wavelength modes

IR modes:

φ : $k < \varepsilon a H$

coarse-grained field (locally FLRW)

INFLATION: Stochastic Approach

Split a quantum fields $(\hat{\Phi}, \hat{\Pi})$ into long and short modes

$$\hat{\Phi} = \varphi + \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, \Theta\left(k - \varepsilon a(t) H\right)\, \hat{\phi}_{\mathbf{k}}(t) e^{i\mathbf{k}.\mathbf{x}}$$

$$\dot{\hat{\Phi}} = \hat{\Pi} = \pi + \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, \Theta\left(k - \varepsilon a(t) H\right)\, \dot{\hat{\phi}}_{\mathbf{k}}(t) e^{i\mathbf{k}.\mathbf{x}}$$

$$\hat{\phi}_k = \frac{\delta\phi_k}{a}$$

time-dependent window function

Ignoring the gradient term

Langevin's equation
$$\begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}N} = \pi + \hat{\xi}_\varphi \\[2ex] \dfrac{\mathrm{d}\pi}{\mathrm{d}N} = -(3 - \epsilon_H)\pi - \dfrac{V_{,\varphi}}{3H^2} + \hat{\xi}_\pi \end{cases}$$

e-folding number: $\mathrm{d}N = H\mathrm{d}t$

We use N as the time variable, hence implicitly work in the **uniform-N gauge**.

INFLATION: Stochastic Approach

For a light scalar field, Qunatum stochastic noises:

$$\hat{\xi}_\varphi(t,\mathbf{x}) = -\frac{\mathrm{d}k_c}{\mathrm{d}t}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\delta(k-k_c)\;e^{i\boldsymbol{k}.\mathbf{x}}\;\hat{\phi}_{\mathbf{k}}(t)\,,$$

$$\hat{\xi}_\pi(t,\mathbf{x}) = -\frac{\mathrm{d}k_c}{\mathrm{d}t}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\delta(k-k_c)\;e^{i\boldsymbol{k}.\mathbf{x}}\;\dot{\hat{\phi}}_{\mathbf{k}}(t)\,,$$

$k_c = \varepsilon a(t) H$

cutoff parameter

commute as $\varepsilon \to 0$

$$\frac{\langle[\hat{\xi}_\varphi,\hat{\xi}_\pi]\rangle}{\langle\hat{\xi}_\varphi,\hat{\xi}_\varphi\rangle} \ll 1$$

and become classic noises:

$$\langle\hat{\xi}_\varphi(N_1)\;\hat{\xi}_\varphi(N_2)\rangle = \left(\frac{H}{2\pi}\right)^2 \delta(N_1-N_2)$$

$$\langle\hat{\xi}_\pi(N_1)\;\hat{\xi}_\pi(N_2)\rangle \sim \mathcal{O}(\varepsilon^4)$$

INFLATION: Stochastic Approach

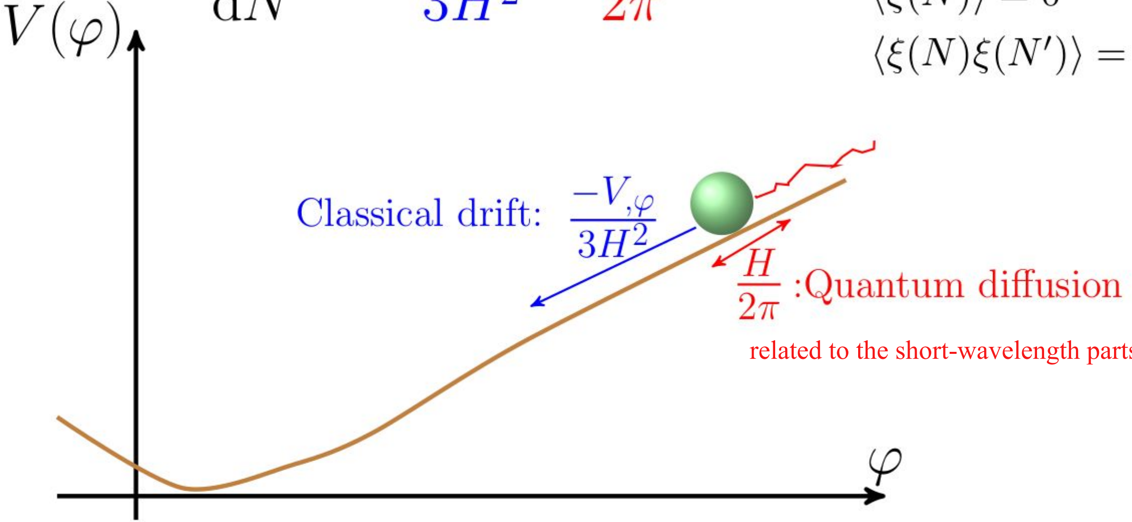
Langevin equation in Slow-roll regime:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}N} = -\frac{V_{,\varphi}}{3H^2} + \frac{H}{2\pi}\xi(N)$$

Normalized white noise

$\langle \xi(N) \rangle = 0$

$\langle \xi(N)\xi(N') \rangle = \delta(N - N')$



related to the short-wavelength parts of the fields

The scalar noise amplitude is independent of the cutoff parameter

STOCHASTIC INFLATION

- A powerful formalism to investigate quantum effects in inflating space times.
- It consists of an **effective theory** for the long-wavelengths part of quantum fields, when coarse-grained at super-Hubble scales.
- Quantum fields thus behave in a **classical**, and evolve according to stochastic Langevin equations.
- In excellent **agreement with QFT** techniques. (Finelli et al. (09), …)
- It can go **beyond perturbative QFT** and describe the non-perturbative evolution of the coarse-grained fields.
- It relies on the **separate universe approach** (Wands et al. (00), …)
- This can be used to reconstruct the primordial density perturbation on super-Hubble scales, by **combining stochastic inflation with the δN formalism** (Fujita et al. (13), …)
- This gives rise to the **stochastic-δN approach**, which has been recently used to derive the full **probability distribution of the primordial density field**, finding large *deviations from a Gaussian statistics* in the nonlinear tail of the distribution (Ezquiaga et al. (20), …)
- …

Stochastic Formalism: gauge fields

Fujita, Obata (17)
A.T. et al. (19, 20, 22)
Fujita et al. (22)

Unified electric and magnetic field using an auxiliary vector field X_i

$$E = X|_{\nu \to n+\frac{1}{2}} \,, \qquad B = X|_{\nu \to n-\frac{1}{2}} \,,$$

Long-Short decomposition:

$$\boldsymbol{X}(t, \boldsymbol{x}) = \boldsymbol{X}^{\mathrm{IR}}(t, \boldsymbol{x}) + \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \Theta\left(k - \varepsilon a H\right) \boldsymbol{X}_{\boldsymbol{k}}(t) \; e^{i\boldsymbol{k}.\boldsymbol{x}}$$

mode function $X(k, \eta) = i\frac{\sqrt{\pi}}{2} \; kH^2 \; \eta^{5/2} \; H_\nu^{(1)}(-k\eta)$

Langevin equation:

$$\boldsymbol{\mathcal{X}}' = b_\nu \; \boldsymbol{\mathcal{X}} + D_\nu(\varepsilon) \; \boldsymbol{\xi} \,, \qquad \boldsymbol{\mathcal{X}} = \frac{\boldsymbol{X}^{IR}}{\sqrt{2\epsilon_H} M_P H}$$

Stochastic Formalism: gauge fields

Langevin equation: $$\boldsymbol{\mathcal{X}}' = b_\nu \, \boldsymbol{\mathcal{X}} + D_\nu(\varepsilon) \, \boldsymbol{\xi}$$

Electromagnetic fields have no classical background values,

$$\boldsymbol{\mathcal{X}}(N) = D_\nu(\varepsilon) \, e^{b_\nu N} \int_0^N e^{-b_\nu s} \mathrm{d}\boldsymbol{W}(s) \,,$$

$$D_\nu(\varepsilon) = \sqrt{6\mathcal{P}_\zeta} \times \begin{cases} \dfrac{2^{|\nu|}}{3} \dfrac{\Gamma(|\nu|)}{\sqrt{2\pi}} \left(1 + \dfrac{|\frac{5}{2} - |\nu||}{Q_\nu}\right) \varepsilon^{-b_\nu} & |\nu| \neq 5/2 \\ 1 & \nu = \pm 5/2 \,. \end{cases}$$

$$b_\nu \equiv |\nu| - \frac{5}{2} + \mathcal{O}(\epsilon_H)$$

Ornstein-Uhlenbeck (OU) process: $b_\nu < 0$

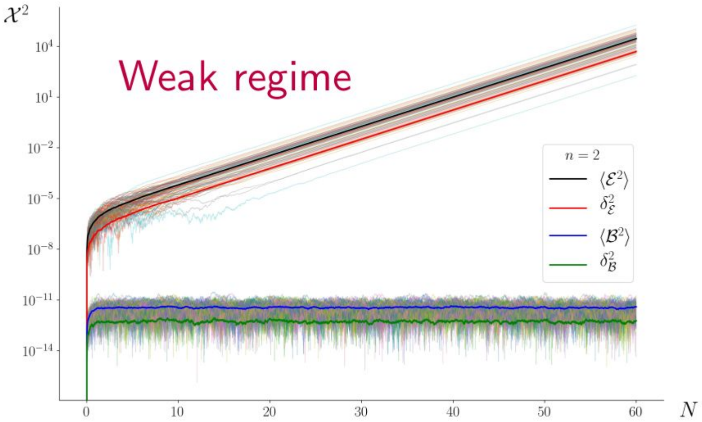
frictional drift force $-|b_\nu|\boldsymbol{\mathcal{X}}$ $\sim$ the random force $D_\nu \boldsymbol{\xi}$

Equilibrium state: $\langle \mathcal{X}^2 \rangle_{\mathrm{eq}} = \dfrac{3D_\nu^2}{2|b_\nu|}$ at around $N_{\mathrm{eq}} \simeq \mathcal{O}(\ln 10/|b_\nu|)$

Stochastic Formalism: gauge fields

$$\mathcal{B}' = -\mathcal{B} + \frac{5\sqrt{\mathcal{P}_\zeta}}{2\sqrt{6}}\,\varepsilon\,\xi$$

$$\mathcal{E}' = -\epsilon_H\,\mathcal{E} + \sqrt{6\mathcal{P}_\zeta}\,\xi$$



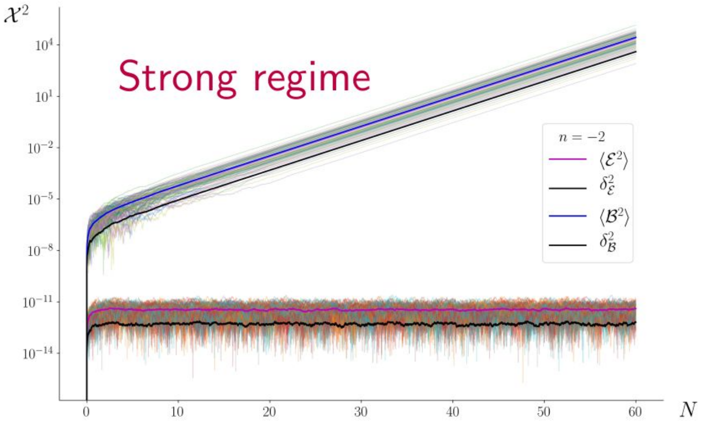
$$N^B_{\rm eq} \simeq \mathcal{O}(1)$$

$$R(N=60) \simeq 10^{-10}$$

$$B_{\rm now} \simeq 10^{-13}\ G$$

$$\mathcal{E}' = -\mathcal{E} + \frac{5\sqrt{\mathcal{P}_\zeta}}{2\sqrt{6}}\,\varepsilon\,\xi$$

$$\mathcal{B}' = -\epsilon_H\,\mathcal{B} + \sqrt{6\mathcal{P}_\zeta}\,\xi$$

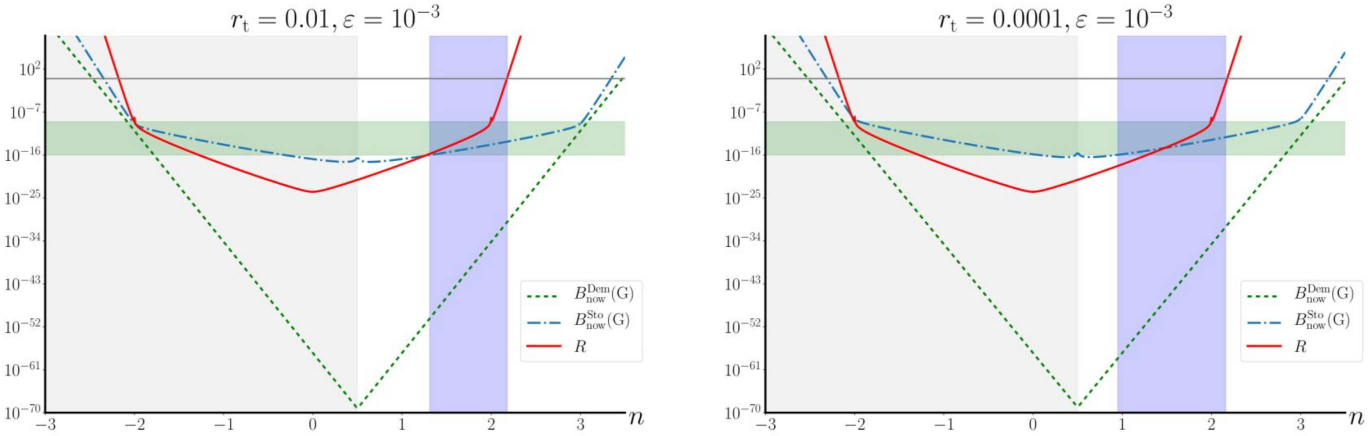


$$N^B_{\rm eq} \simeq \mathcal{O}(10^3)$$

$$R(N=60) \simeq 10^{-10}$$

$$B_{\rm now} \simeq 10^{-9}\ G$$

Stochastic Formalism: gauge fields



Gray region: Strong regime

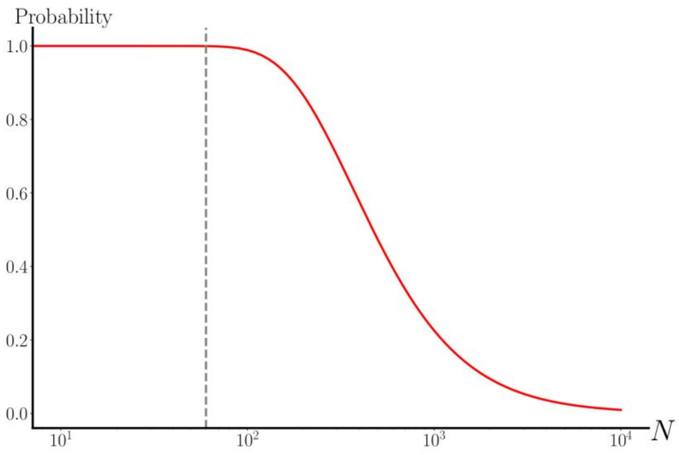
Green band: $10^{-9}G \gtrsim B_{\rm obs} \gtrsim 10^{-16}G$

Blue band: Healthy parameter space

Stochastic Formalism: gauge fields

Wiener Process

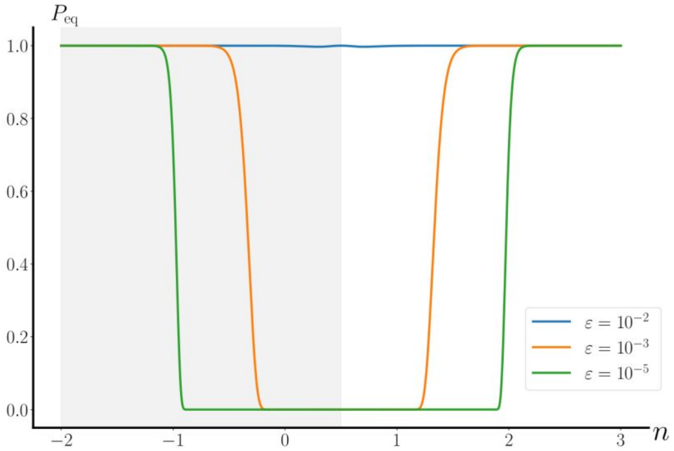
$$P(10^{-16} \lesssim B_{\rm now} \lesssim 10^{-9}; N)$$



$n = 3$ and $n = -2$

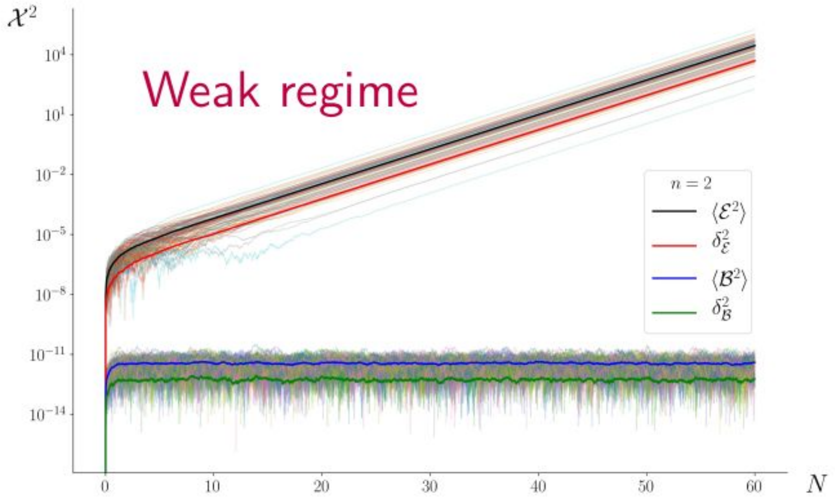
OU Process

$$P_{\rm eq}(10^{-16} \lesssim B_{\rm now} \lesssim 10^{-9})$$



$-2 < n < 3$

Stochastic Formalism: gauge fields



$n = 2.2$

$$\mathrm{d}\boldsymbol{\mathcal{B}} = -0.8\ \boldsymbol{\mathcal{B}}\ \mathrm{d}N + 1.14\sqrt{\mathcal{P}_\zeta}\ \varepsilon^{4/5}\ \mathrm{d}\boldsymbol{W}$$

$$\mathrm{d}\boldsymbol{\mathcal{E}} = +0.2\ \boldsymbol{\mathcal{E}}\ \mathrm{d}N + 3.4\sqrt{\mathcal{P}_\zeta}\ \varepsilon^{-1/5}\ \mathrm{d}\boldsymbol{W}$$

$N_{\rm vio} \simeq 55$

$$B_{\rm now} \simeq 10^{-13}\ \mathrm{G}$$

$n = 2$

$$\mathrm{d}\boldsymbol{\mathcal{B}} = -\boldsymbol{\mathcal{B}}\ \mathrm{d}N + \frac{5\sqrt{\mathcal{P}_\zeta}}{2\sqrt{6}}\ \varepsilon\ \mathrm{d}\boldsymbol{W}\,.$$

$$\mathrm{d}\boldsymbol{\mathcal{E}} = -\epsilon_H\ \boldsymbol{\mathcal{E}}\ \mathrm{d}N + \sqrt{6\mathcal{P}_\zeta}\ \mathrm{d}\boldsymbol{W}$$

$N_{\rm eq} \simeq 3 \qquad R \simeq 10^{-10} \ll 1$

SOME THOUGHTS

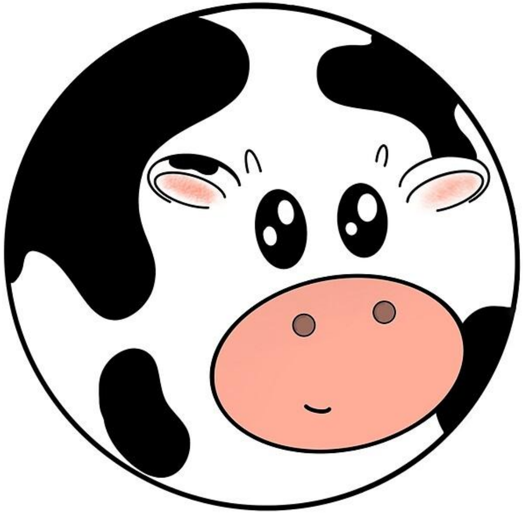
- The stochastic formalism consists of an effective theory for IR modes of the quantum fields, which are coarse grained at a fixed physical scale.

- The amplitude of the electromagnetic noises, in $f^2 F_{\mu\nu} F^{\mu\nu}$ models in which $f \propto \eta^n$, depends on the cutoff parameter ε and n which indicates the scale dependency of the electromagnetic fields spectra.
- The main reason for the amplification of the magnetic field in this case is due to the Ornstein-Uhlenbeck process which settles the fields into an equilibrium state and prevent them from decaying.

- Stochastic effects from massive fields during inflation.
- Stochastic analysis of the non-Gaussian noises.

MY THOUGHTS

Modified Spherical Cow:
A cow with a tail



IPM Cosmic Magnetism Group

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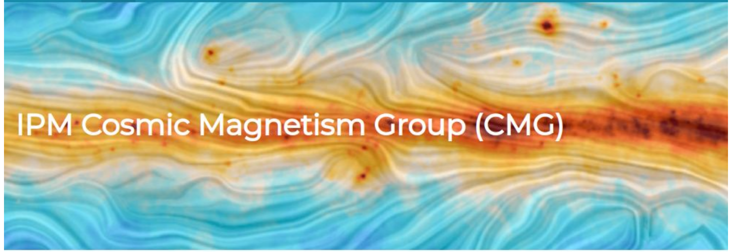


Origins, evolution, and effects of the cosmic magnetic field on structure formation in the universe are among pressing questions in astrophysics and cosmology. School of Astronomy (SoA) at Institute of Research in Fundamental Sciences (IPM) hosts researchers who are actively involved in both observational and theoretical studies of cosmic magnetism on various scales. For a better understanding of the subject it is very insightful to connect theoreticians with observers. The **Cosmic Magnetism Group (CMG)** is hence created to define collaborative projects.

Cosmic Magnetism Group Meeting (CMGM) is the monthly meetings of the CMG to

1- define and progress collaborative projects

2- present the latest findings in the field through journal clubs and free discussions

3- host talks from famous external and international scientists in the field

It is held every **Sunday** at **10 A.M.** (GMT+03:30) Iran Time in the SoA Seminar Room. (Location)

The CMGM are now either in-person or hybrid.

If you would like to attend virtually or in person, please get in touch with the organizers.

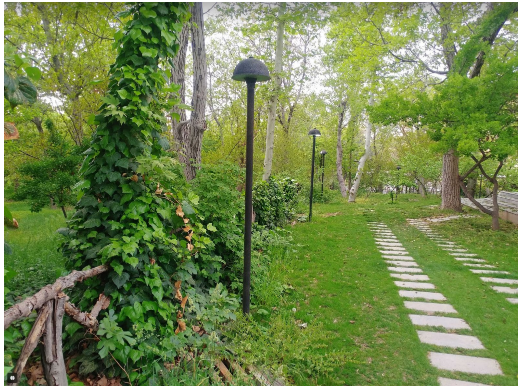
Organizer(s):

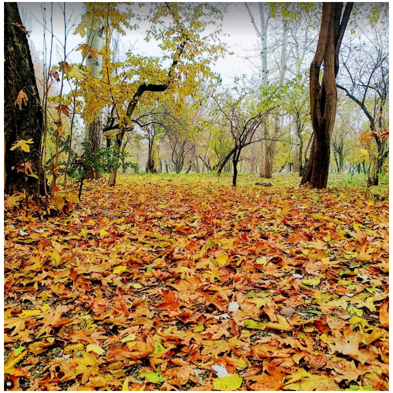
Alireza Talebian (talebian@ipm.ir)

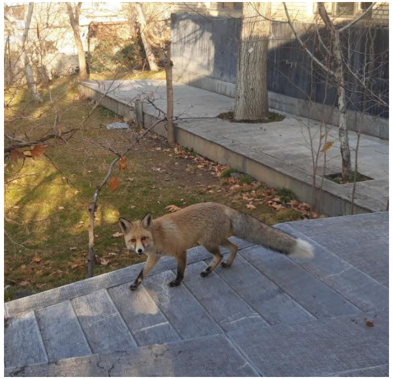
List of Talks:

- **Spring 2024 (1403)**

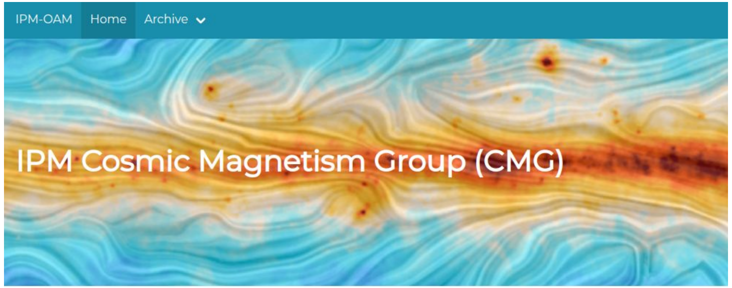
Date	Topic(s)
23 Apr (4 Ordibehesht)	Inaugural Meeting







IPM Cosmic Magnetism Group



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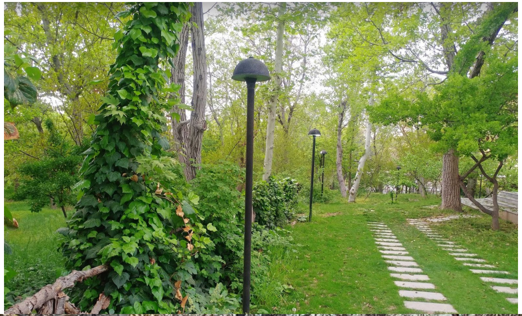
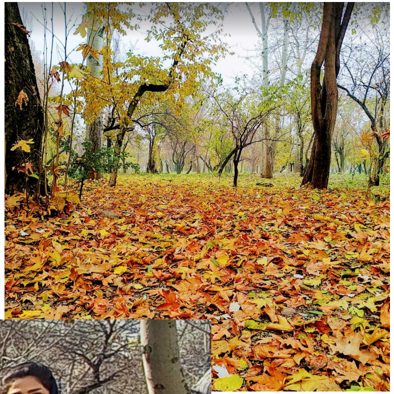
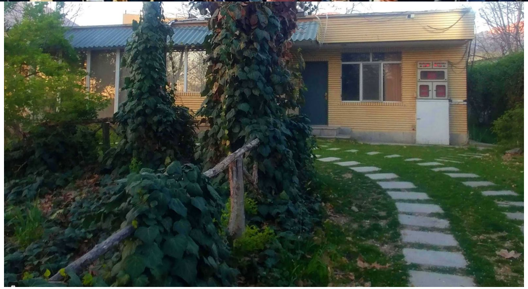
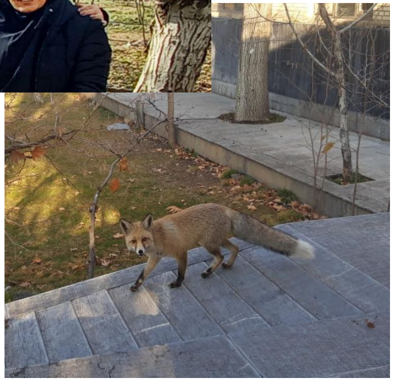
23 Apr (4 Ordibehesht)	Inaugural Meeting

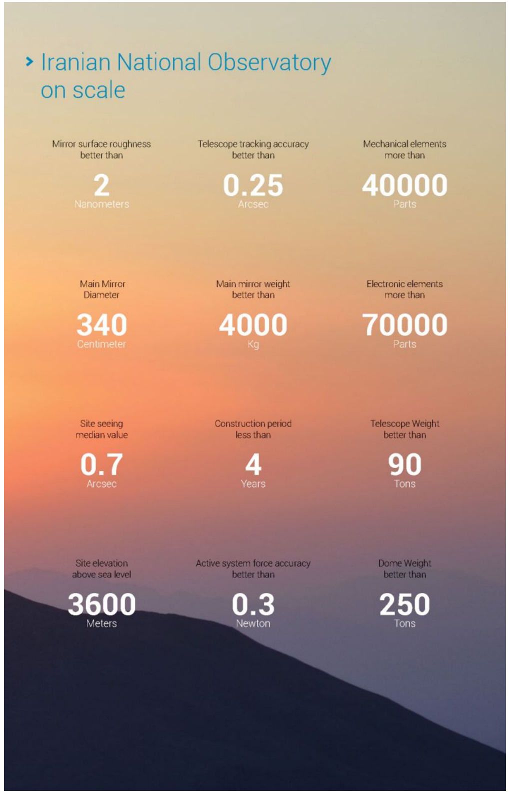


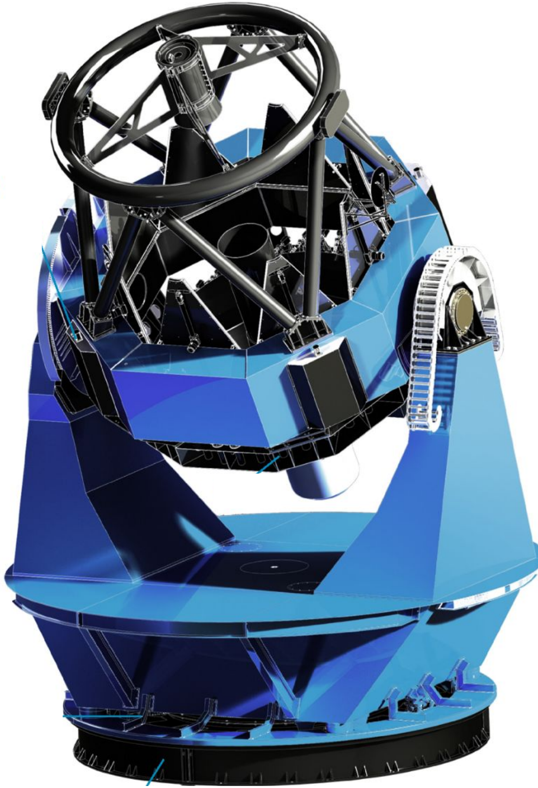






Iranian National Observatory

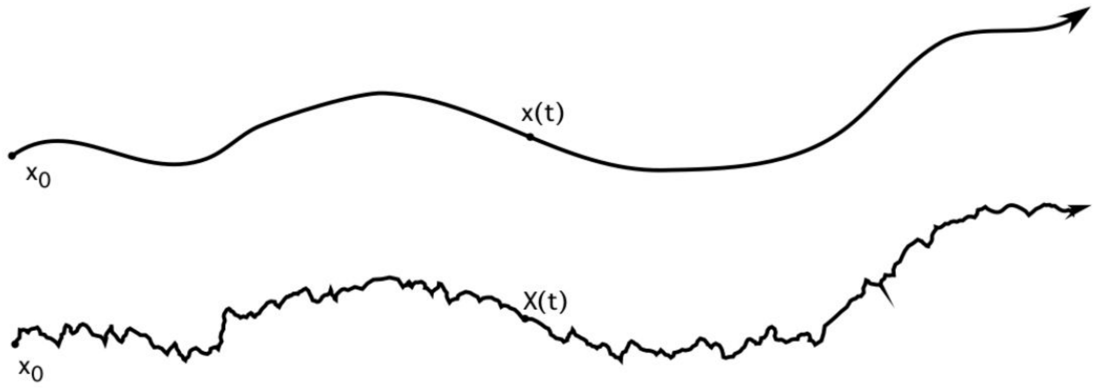




A 1-min Clip before lunch!

Backup slides

Stochastic Differential Equations: Wiener Process



stochastic process $X(\cdot)$: solution of SDE

$$\begin{cases} \dot{X}(t) = b(X(t)) + B(X(t))\, \xi(t)\,, & (t > 0) \\ X(0) = X_0, & \end{cases}$$

White noise:

$$\langle \xi(t)\, \xi(t') \rangle = \delta(t - t')\,.$$

Stochastic Differential Equations: Wiener Process

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White noise:

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$$X(t) = X_0 + \int_0^t b(X,s)\, \mathrm{d}s + \int_0^t B(X,s)\, \mathrm{d}W$$

Wiener Process (Brownian motion):

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \xi(t)$$

$$\langle W(t) \rangle = 0\,, \quad \langle W(t)W(s) \rangle = min\{t,s\}\,, \quad \langle W^2(t) \rangle = t\,,$$

Example of SDE

Examples

- Wiener process (Brownian motion)
- Stock prices (geometric Brownian motion)
- Brownian bridge
- Ornstein–Uhlenbeck process (mean-reverting process)

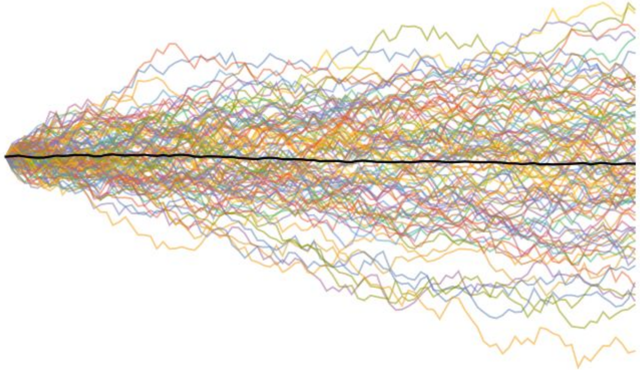

Figure: Wiener

$$\mathrm{d}X_{\mathrm{t}} = \sigma \, \mathrm{d}W_{\mathrm{t}}$$

σ : *constant*.

Example of SDE

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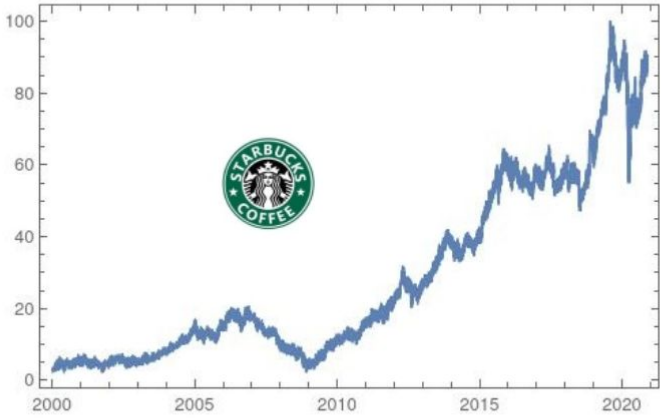

Figure: Starbucks Corporation (SBUX) Stock Price

$$\begin{cases} \mathrm{d}S_{\mathrm{t}} = \mu \; S_{\mathrm{t}} \; \mathrm{d}t + \sigma \; S_{\mathrm{t}} \; \mathrm{d}W_{\mathrm{t}} \,, \\ S(0) = s_0, \end{cases}$$

$$S(t) = s_0 e^{\sigma W(t) + \left(\mu - \frac{\sigma^2}{2}\right) t}$$

$\mu > 0$: Drift $\quad \sigma$: volatility

Example of SDE

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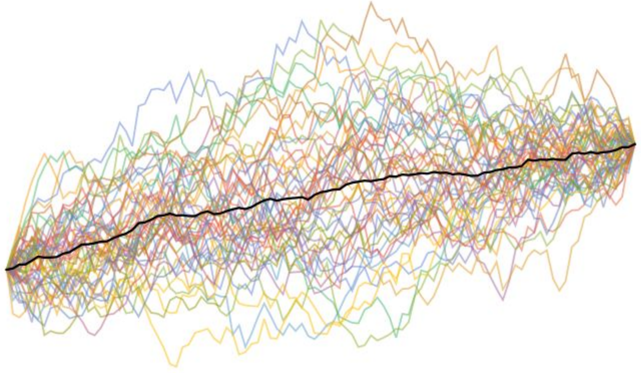

$$\begin{cases} \mathrm{d}B_{\mathrm{t}} = -\dfrac{B_{\mathrm{t}}}{1-t}\,\mathrm{d}t + \mathrm{d}W_{\mathrm{t}}\,, \\ B(0) = 0, \end{cases}$$

$$B(t) = (1-t)\int_0^t \frac{1}{1-s}\mathrm{d}W_{\mathrm{s}}$$

Example of SDE

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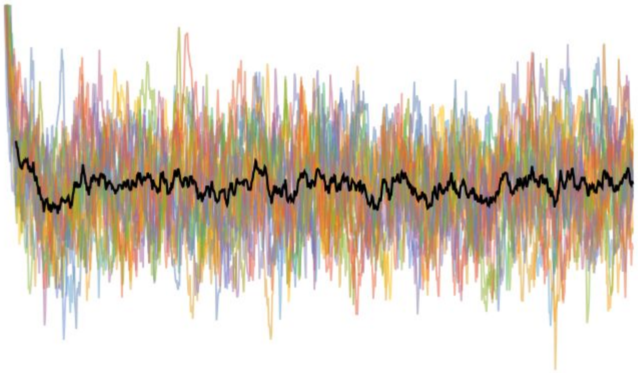


$$\begin{cases} \mathrm{d}X_{\mathrm{t}} = -b\, X_{\mathrm{t}}\, \mathrm{d}t + \sigma\, \mathrm{d}W_{\mathrm{t}}\,, \\ X(0) = X_0, \end{cases}$$

$$X(t) = X_0\, e^{-b\,t} + \sigma \int_0^t e^{-b(t-s)}\, \mathrm{d}W$$

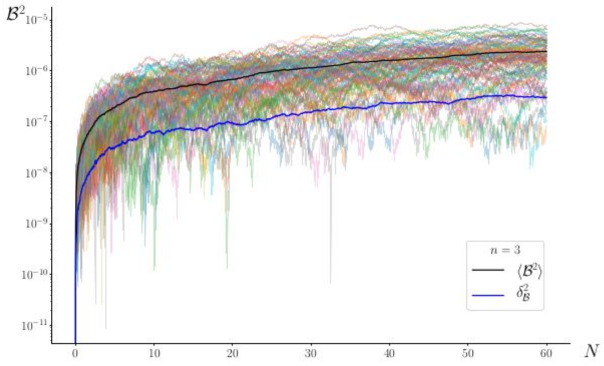
Friction $b > 0$, diffusion: σ

Stochastic effects from helical noises!

Stochastic Formalism: Helical Gauge Fields

$$\mathcal{L}_{\rm EM} = -\frac{I^2(\eta)}{4}\left(F^{\mu\nu}F_{\mu\nu} + \gamma F^{\mu\nu}\tilde{F}_{\mu\nu}\right)\,, \qquad I(\eta) \propto a^{-n}$$

The gauge field undergoes tachyonic growth of one of polarizations and leads to generation of a helical magnetic field.



$$\mathcal{B}' = -(2+n)\mathcal{B} + D_B \xi$$
$$\mathcal{E}' = -(2-n)\mathcal{E} - 2\xi\mathcal{B} + D_E \xi$$

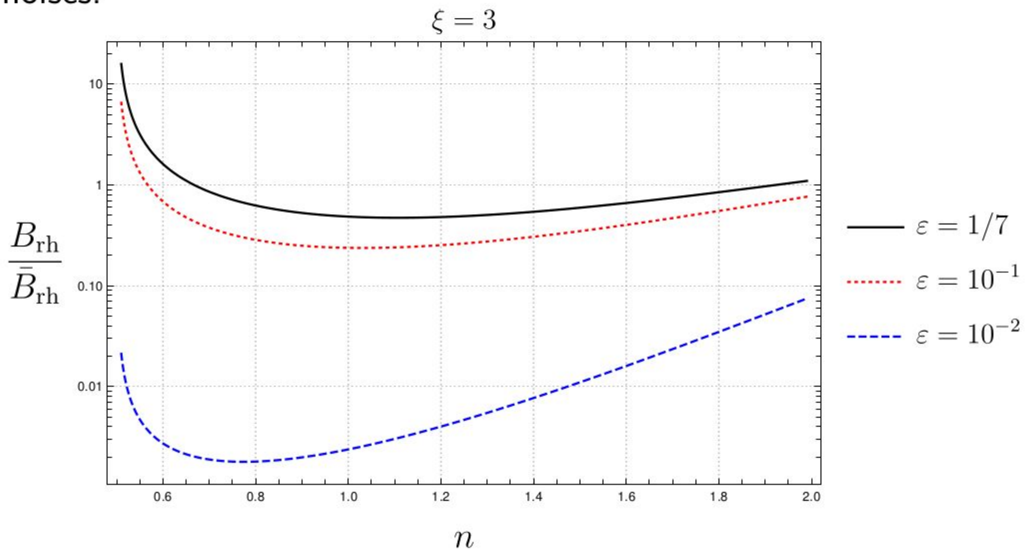
$$\xi = -n\gamma$$

$$D_B \simeq \frac{e^{\pi\xi}\sqrt{\xi}}{\pi\sqrt{3\pi}}\,\frac{\Gamma(2n-1)}{2^n}\,\frac{H}{M_{\rm Pl}}\varepsilon^{3-n}$$

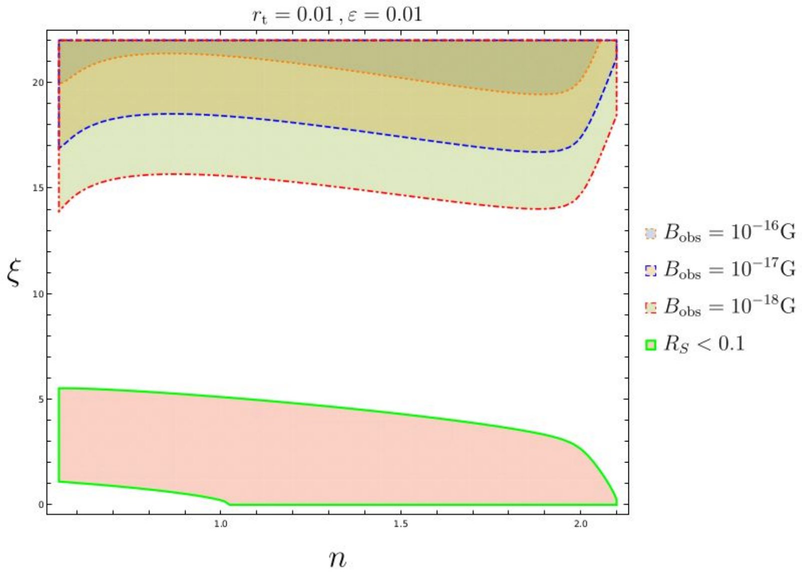
$$D_E \simeq D_B\frac{(2n-1)}{\varepsilon}$$

Stochastic Formalism: Helical Gauge Fields

Tachyonic growth of the magnetic fields are replaced by a mean-reverting process of stochastic dynamics. As a result, the magnetic fields settle down into an equilibrium state with the amplitude significantly smaller than what is obtained in the absence of the stochastic noises.



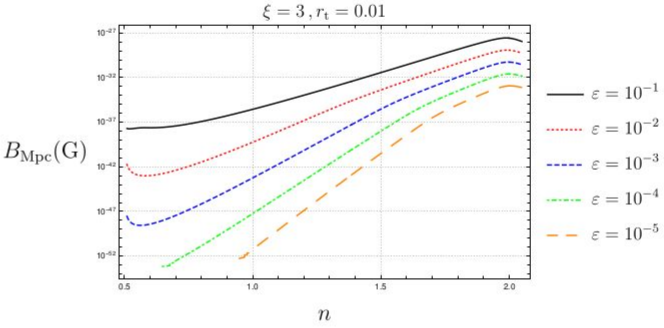
Stochastic Formalism: Helical Gauge Fields

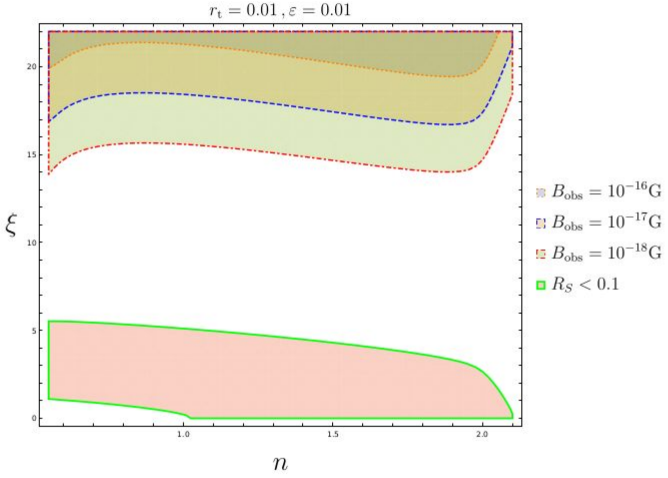


Back-reaction parameter: R_S

Stochastic Formalism: Helical Gauge Fields

Helicity conservation: $B^2 L \propto a^{-3}$

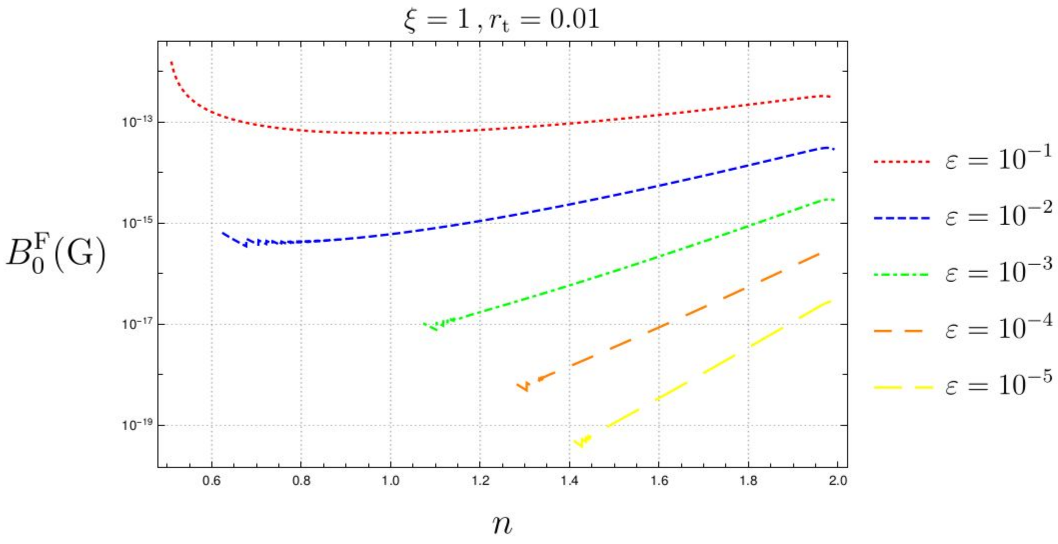




$$R_S = \frac{n}{6\epsilon_\phi}\left(\mathcal{E}^2 - \mathcal{B}^2 - 2\gamma\mathcal{E}\cdot\mathcal{B}\right) \ll 1$$

Stochastic Formalism: Helical Gauge Fields

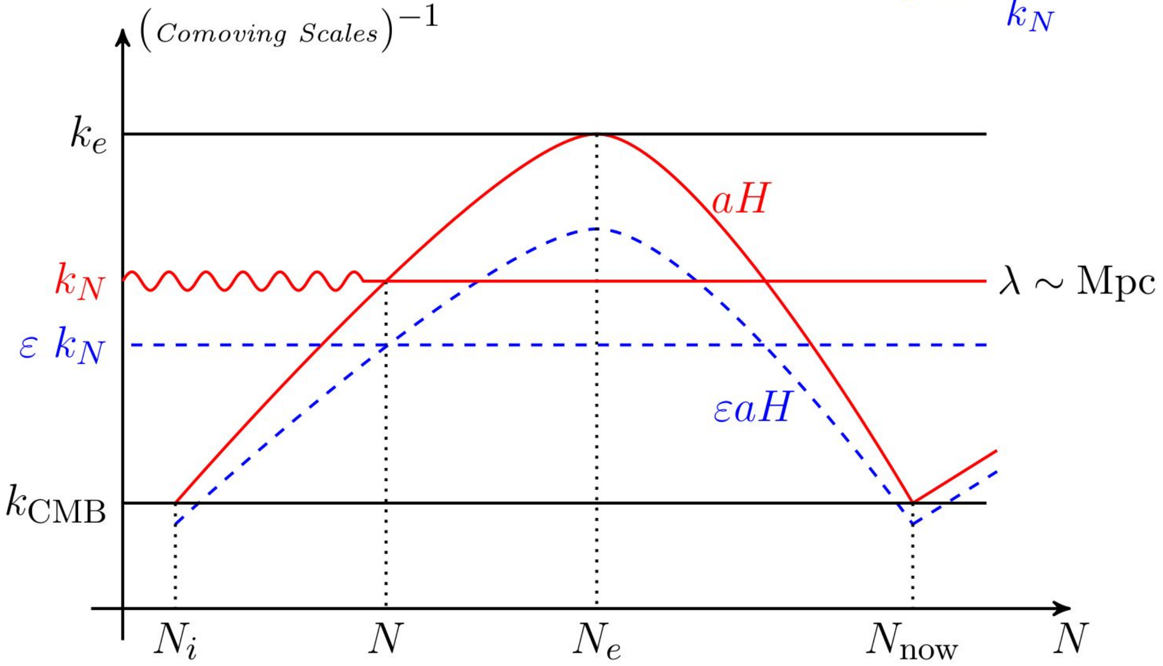
Flux conservation: $B^2 \propto a^{-4}$



Stochastic Formalism: gauge fields

How small ε can be?

$$\varepsilon \equiv \frac{k_{\rm CMB}}{k_N}$$



$$k_{\rm CMB} = \left(0.05 - 10^{-4}\right)\ {\rm Mpc}^{-1} \Longrightarrow \quad \varepsilon \simeq \mathcal{O}\left(10^{-2} - 10^{-5}\right)$$